April 3, 2002 WOCOMAL VARSITY MEET

ROUND I: Elementary number theory

NO CALCULATOR USE

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. If N written in base 2 is 11000, what is the integer one less than N, written in base 2?

2. Let P be a prime number other than 5. Find the sum of the positive factors of 5P in terms of P. Include 1.

3. What is the smallest number that leaves a remainder of 9 when divided by 10, a remainder of 8 when divided by 9, a remainder of 7 when divided by 8, . . . , a remainder of 2 when divided by 3, and a remainder of 1 when divided by 2?

ANSWERS

1. (1 pt) ______________________

2. (2 pts) ______________________

3. (3 pts) ______________________

St.John's, South

ROUND II: Algebra 1 - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find the value of k for which (-1.5, 2.5) is a point on the graph of $y = 2x^2 + 2x + k$.

2. Solve for x. $(x+2)(2x-1)+(x-3)(2x-1)-(3x+5)(2x-1)=0$

3. If $x + y = 11$ and $y = \frac{15}{x}$, find the value of $x^2 + y^2$.

ANSWERS

1. (1 pt) ____________

2. (2 pts) ____________

3. (3 pts) ____________

Leicester, Hudson, St.Peter-Marian

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ROUND III: Open geometry

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. If the sum of the degree measures of all interior angles except one of a convex polygon is 2550 degrees, find the measure of the other angle.

2. A certain chord of a circle is 6 inches long and is the perpendicular bisector of a radius of the circle. Determine the area of the circle, in terms of π.

3. In trapezoid ABCD, E and F are the midpoints of legs $\overline{AB}$ and $\overline{CD}$ respectively. $\overline{CA}$ intersects $\overline{EF}$ at G and $\overline{BD}$ intersects $\overline{EF}$ at H. If BC = 15 and AD = 20, what is GH?

ANSWERS

1. (1 pt) ______________

2. (2 pts) ______________

3. (3 pts) ______________

Burncoat, Northbridge, Southbridge

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ROUND IV: Logs, exponents, radicals

NO CALCULATOR USE

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Perform the indicated operations: $\left(4\sqrt{20}+\sqrt{80}\right)\div\left(2\sqrt{20}\right)$

2. Solve for x: $27^{2x-2}=9^{x+5}$

3. If both m and n are bigger than 1 and for all positive numbers x, $\log_n x = 3\log_m x$, write an equation expressing m explicitly in terms of n.

ANSWERS

1. (1 pt) ____________________

2. (2 pts) ____________________

3. (3 pts) ____________________

Douglas, Hudson, Tahanto

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ROUND V: Trigonometry - open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find $\sin\theta$ when the terminal side of θ is in quadrant IV and $\tan\theta = \frac{-4}{3}$.

2. If $y = \sin\theta + 4$, for $0^\circ \le \theta \le 180^\circ$, what are the maximum and minimum values of y?

3. Express in terms of x: $\sin\left[2\cos^{-1}(-x)\right]$

ANSWERS

1. (1 pt) ____________

2. (2 pts) max ________ min ________

3. (3 pts) ________________________

South, Tahanto, Westboro

TEAM ROUND: Topics of previous rounds and open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM (except in number 3.)AND ON THE SEPARATE TEAM ANSWER SHEET 2 points each

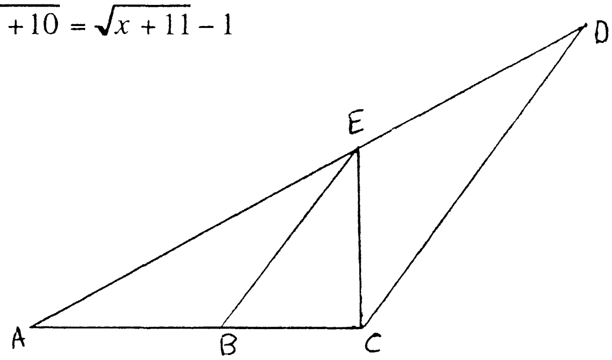
1. How many positive integers are factors of 60 or factors of 150, but are not factors of 210?

2. If a two-digit integer is k times the sum of its digits, the number formed by interchanging the digits is the sum ot the digits multiplied by what (in terms of k)?

3. The minute hand and the hour hand of a clock are perpendicular to each other twice between 5.00 and 6:00. Compute the elapsed time, correct to the nearest second, between the two times.

4. Find all real solutions of $\sqrt{3x+10} = \sqrt{x+11} - 1$

5. Given: $\overline{BE} \parallel \overline{CD}$
   $\overline{EC} \perp \overline{AC}$
   $AE = 24,\ AB = 10$
   $\cos A = \dfrac{7}{8}$
   Find CD



6. If $x@y = \dfrac{xy}{x+y}$, find and simplify: $A @\left(A@\dfrac{1}{A}\right) - (A @ A) @\dfrac{1}{A}$

7. Q is what percent of 20% of 15? Answer in terms of Q.

8. Specify without absolute values all real numbers x for which $|5 - |x|| < 14$. Do not answer with a graph.

9. How many 3-digit positive integers are divisible by 11?

Bromfield, Doherty, Quaboag, St..John's, Shrewsbury, Tahanto, Tantasqua, Worcester Academy

April 3, 2002 NOCOMAL Varsity Meet ANSWERS

ROUND I — # thry

1. 1 pt 10111
2. 2 pts $6P+6$ or $6(P+1)$
3. 3 pts 2519

ROUND II — Alg 1

1. 1 pt 1 $[k=1]$
2. 2 pts $\frac{1}{2}, -6$ $[x = \quad]$
3. 3 pts 91

ROUND III — geom

1. 1 pt 150°
2. 2 pts 12π sq in (may omit units)
3. 3 pts $2\frac{1}{2}$ or 2.5 or $\frac{5}{2}$

ROUND IV — logs exp rad

1. 1 pt 3
2. 2 pts 4 $[x=4]$
3. 3 pts $m = n^3$

ROUND V — trig

1. 1 pt $-\frac{4}{5}$ or $-.8$
2. 2 pts max 5 min 4 Need both
3. 3 pts $-2x\sqrt{1-x^2}$ or $-2x(1-x^2)^{\frac{1}{2}}$

TEAM ROUND 2 pts each

1. 8
2. $11-k$
3. 32 min 44 sec
4. -2
5. 33.6 or $33\frac{3}{5}$
6. $\cancel{\frac{A^3-A^2}{A^3+2A^2+2A+4}}$ 0 ~~or~~ $\cancel{\frac{A^2(A-1)}{(A+2)(A^2+2)}}$ ~~OR top of one and bottom of other~~
7. $\frac{100Q}{3}$ % or $33.\bar{3}Q$ %
8. $-19 < x < 19$
9. 81

April 3, 2002 WOCOMAL Varsity Meet BRIEF SOLUTIONS

ROUND I

1. $\begin{array}{r} 11000 \\ -\quad 1 \\ \hline 10111 \end{array}$ $\begin{array}{l} = 24_{10} \\ \\ = 23_{10} \end{array}$

2. $1+5+P+5P = 6+6P$

3. One more than that number is divisible by 10, 9, 8, 7, … 3, 2. The LCM of these nine numbers is $2^3 3^2 5.7 = 2520$. The number sought is 2519

ROUND II

1. $2.5 = 2\left(-\frac{3}{2}\right)^2 + 2\left(-\frac{3}{2}\right) + k$

$\frac{5}{2} = \frac{9}{2} - \frac{6}{2} + k \Rightarrow K = 1$

2. Common factor

$(2x-1)[x+2+x-3-(3x+5)] = 0$

$(2x-1)(-x-6) = 0$

$x = \frac{1}{2}$ or -6

3. $(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^2 - 2xy = x^2 + y^2$

$121 - 2 \cdot 15 = x^2 + y^2$

$91 = x^2 + y^2$

ROUND III

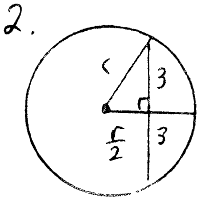
1. Sum of all ∡ meas $= (n-2)180°$

$\frac{2550°}{180°} = 14\frac{1}{6}$. For $n-2 = 15$ we get

2700° for all 17 angles and

$2700° - 2550° = 150°$ for the other angle

2.



$\left(\frac{r}{2}\right)^2 + 3^2 = r^2$

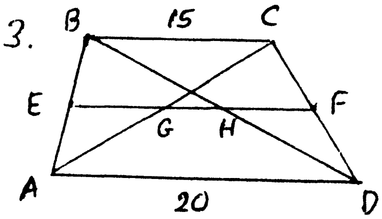
$\frac{r^2}{4} + 9 = r^2$

$r^2 = 12$

sq in

ROUND III cont.

3.



$EF = \frac{1}{2}(15+20)$

$EH = \frac{1}{2}AD = 10$

$GF = \frac{1}{2}AD = 10$

$EH + GF = EG + GH + GH + HF$

$= (EG + GH + HF) + GH$

$= EF + GH$

$10 + 10 = 17.5 + GH \Rightarrow GH = 2.5$

ROUND IV

1. $\frac{4\sqrt{20} + \sqrt{80}}{2\sqrt{20}} = \frac{4\sqrt{20}}{2\sqrt{20}} + \frac{\sqrt{80}}{2\sqrt{20}}$

$= 2 + \frac{\sqrt{4}}{2} = 2 + 1. = 3$

2. $27^{2x-2} = 9^{x+5}$

$3^{6x-6} = 3^{2x+10}$

$6x - 6 = 2x + 10$

$4x = 16 \Rightarrow x = 4$

3. First $3\log_m x = \log_m x^3$

Let $\log_n x = \log_m x^3 = y$

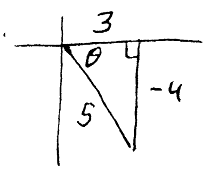
Then $n^y = x$ and $m^y = x^3$

$\therefore (n^y)^3 = m^y$

$(n^3)^y = m^y \Rightarrow n^3 = m$

ROUND V

1.



$\sin\theta = \frac{y}{r} = \frac{-4}{5}$

2. For $0° \le \theta \le 180°$, $0 \le \sin\theta \le 1$

Max $y = 1 + 4 = 5$

Min $y = 0 + 4 = 4$

ROUND V

3. Use $\sin 2\theta = 2\sin\theta\cos\theta$

and $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(-x)^2}$

$$\sin 2\theta = 2\sqrt{1-x^2}\,(-x)$$
$$= -2x\sqrt{1-x^2}$$

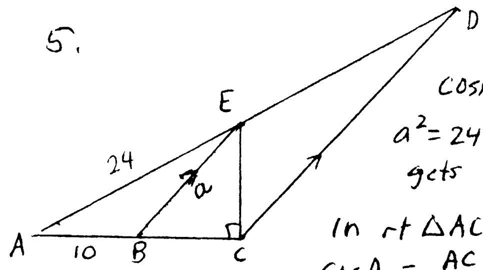
4. $\sqrt{3x+10} = \sqrt{x+11} - 1$

$$3x+10 = x+11 - 2\sqrt{x+11} + 1$$
$$2x-2 = -2\sqrt{x+11}$$
$$x-1 = -\sqrt{x+11}$$
$$x^2-2x+1 = x+11$$
$$x^2-3x-10 = 0$$
$$(x+2)(x-5) = 0$$

$x=-2$ ✓

$x = 5$ doesn't check

5.



Cosine law in ΔABE

$a^2 = 24^2+10^2-2\cdot 24\cdot 10\cdot\frac{7}{8}$

gets $a = 16$

In rt ΔACE

$\cos A = \frac{AC}{AE}$ so $\frac{7}{8} = \frac{AC}{24}$

which gets $AC = 21$.

$\Delta ABE \sim \Delta ACD$ gets $\frac{AB}{AC} = \frac{BE}{CD}$ or $\frac{10}{21} = \frac{16}{CD}$

and $CD = 33.6$

6. $A @ \frac{1}{A} = \frac{1}{A+\frac{1}{A}} = \frac{A}{A^2+1}$

$$A @ \frac{A}{A^2+1} = \frac{\frac{A^2}{A^2+1}}{A+\frac{A}{A^2+1}} = \frac{A^2}{A^3+2A} = \frac{A}{A^2+2}$$

$$A @ A = \frac{A^2}{2A} = \frac{A}{2}$$

$$\frac{A}{2} @ \frac{1}{A} = \frac{\frac{1}{2}}{\frac{A}{2}+\frac{1}{A}} = \frac{A}{A^2+2}$$

subtracting gets

0

7. $Q = \frac{x}{100}\cdot\frac{20}{100}\cdot 15 = \frac{3x}{100} \Rightarrow x = \frac{100Q}{3}$

8. $-14 < 5-|x| < 14$

$$-19 < -|x| < 9$$
$$19 > \underbrace{|x| > -9}_{\text{always}} \qquad \therefore -19 < x < 19$$

9 Find out how many multiples of 11 there are
First is 110. Last is 990. They form an arithmetic sequence of n terms

$$t_n = t_1 + (n-1)d$$
$$990 = 110 + (n-1)11$$
$$880 = 11(n-1)$$
$$80 = n-1 \Rightarrow n = 81$$

TEAM ROUND

1. $60 = 2^2\cdot 3\cdot 5$

$150 = 2\cdot 3\cdot 5^2$

$210 = 2\cdot 3\cdot 5\cdot 7$

The factors desired include 2^2 or 5^2

$2^2 = 4$	$5^2 = 25$
$2^2\cdot 3 = 12$	$5^2\cdot 2 = 50$
$2^2\cdot 5 = 20$	$5^2\cdot 3 = 75$
$2^2\cdot 3\cdot 5 = 60$	$5^2\cdot 2\cdot 3 = 150$

8 pos integers

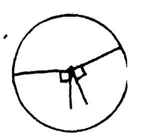
2. $10u+t = K(u+t)$

If $10t+u = x(u+t)$,

then $11(t+u) = (K+x)(u+t)$

$$11 = K+x \Rightarrow x = 11-K$$

3.



Between the two $\perp$ hands times, the hour hand rotates $x°$ and the minute hand rotates $90° + x° + 90°$.

Rotation rates:

minute hand : $\frac{360°}{hr} = \frac{6°}{min}$

hour hand : $\frac{30°}{hr} = \frac{\frac{1}{2}°}{min}$

Let t = time sought, minutes

$$\left.\begin{aligned} 180 + x &= 6t \\ x &= \tfrac{1}{2}t \end{aligned}\right\}\text{ rotation amounts}$$

$$180 = 5\tfrac{1}{2}t$$
$$t = 32.\overline{72}\text{ min} \approx 32\text{ min } 44\text{ sec}$$
$$(.\overline{72}(60) \approx 44)$$